

# Matricvariate and matrix multivariate Pearson type II distributions

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## Abstract

This paper proposes a unified approach to enable the study of diverse distributions in the real, complex, quaternion and octonion cases, simultaneously. In particular, the central, nonsingular matricvariate and matrix multivariate Pearson type II distribution, beta type I distributions and the joint density of the singular values are obtained for real normed division algebras.

## 1 Introduction

In the last twenty years, the concept and the statistical and mathematical techniques known as *multivariate analysis* have changed dramatically. New statistical and mathematical tools for the analysis of multivariate data have been developed in diverse areas of knowledge, and have promoted new disciplines such as pattern recognition, nonlinear multivariate analysis, data mining, manifold learning, generalised multivariate analysis, latent variable analysis and shape theory, among others. These and other fields constitute what is known as *modern multivariate analysis*.

Renewed interest in multivariate analysis in the complex case has emerged in diverse areas, see Metha [26], Ratnarajah *et al.* [30] and Micheas *et al.* [27], among many others. Similarly, several works involving multivariate analysis have appeared in the context of the quaternion case, see Bhavsar [2], Forrester [15], Li and Xue [25], among others. Although receiving little attention from a practical standpoint, but equally interesting from the theoretical point of view, some results have appeared in the octonion case, see Forrester [15]. This lack of widespread interest may be, because as asserted by Baez [1], *...there is still no proof that the octonions are useful for understanding the real world*. We can only hope that eventually this question will be settled one way or another.

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**Key words.** Matricvariate; elliptical distribution; inverted  $T$  distribution; nonsingular central distributions; real, complex, quaternion and octonion random matrices; beta type I distributions.  
2000 Mathematical Subject Classification. Primary 60E05, 62E15; secondary 15A52

In terms of concepts, definitions, properties and notation from abstract algebra, it is possible to propose a unified approach that enables the simultaneous study of the distribution of a random matrix in real, complex, quaternion and octonion cases, which is termed as the distribution of a random matrix for real normed division algebras.

In the real case, the matricvariate Pearson type II distribution appears in the frequentist approach to normal regression as the distribution of the Studentised error, see Díaz-García and Gutiérrez-Jáimez [4] and Kotz and Nadarajah [24]. In Bayesian inference, the matricvariate Pearson type II distribution is assumed as the sampling distribution; then, considering a noninformative prior distribution, the posterior distribution and marginal distributions, the posterior mean and generalised maximum likelihood estimators of the parameters involved are found, Fang and Li [13]. A very important question is that of the role of the Pearson type II distribution in multivariate analysis, because if the matrix  $\mathbf{R}$  has a matricvariate Pearson type II distribution, then the matrix  $\mathbf{R}\mathbf{R}^*$  (or  $\mathbf{R}^*\mathbf{R}$ ) is distributed as beta type I; and the distribution of the latter, in particular, plays a fundamental role in the MANOVA model, see Khatri [21, 23] and Muirhead [28].

The present article is organised as follows; a minimal number of concepts and the notation of abstract algebra and Jacobians are summarised in Section 2. Section 3 then derives the nonsingular central matricvariate Pearson type II and the beta type I distributions and some basic properties. Similarly, results are obtained for the matrix multivariate Pearson type II and the corresponding beta type I distributions, see Section 4. Finally, the joint densities of the singular values are derived in Section 5. We emphasise that all these results are found for real normed division algebras.

## 2 Preliminary results

A detailed discussion of real normed division algebras may be found in Baez [1] and Gross and Richards [16]. For convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

For our purposes, a **vector space** is always a finite-dimensional module over the field of real numbers. An **algebra**  $\mathfrak{F}$  is a vector space that is equipped with a bilinear map  $m : \mathfrak{F} \times \mathfrak{F} \rightarrow \mathfrak{F}$  termed *multiplication* and a nonzero element  $1 \in \mathfrak{F}$  termed the *unit* such that  $m(1, a) = m(a, 1) = 1$ . As usual, we abbreviate  $m(a, b) = ab$  as  $ab$ . We do not assume  $\mathfrak{F}$  associative. Given an algebra, we freely think of real numbers as elements of this algebra via the map  $\omega \mapsto \omega 1$ .

An algebra  $\mathfrak{F}$  is a **division algebra** if given  $a, b \in \mathfrak{F}$  with  $ab = 0$ , then either  $a = 0$  or  $b = 0$ . Equivalently,  $\mathfrak{F}$  is a division algebra if the operation of left and right multiplications by any nonzero element is invertible. A **normed division algebra** is an algebra  $\mathfrak{F}$  that is also a normed vector space with  $\|ab\| = \|a\|\|b\|$ . This implies that  $\mathfrak{F}$  is a division algebra and that  $\|1\| = 1$ .

There are exactly four normed division algebras: real numbers ( $\mathbb{R}$ ), complex numbers ( $\mathbb{C}$ ), quaternions ( $\mathbb{H}$ ) and octonions ( $\mathbb{O}$ ), see Baez [1]. We take into account that  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  and  $\mathbb{O}$  are the only normed division algebras; moreover, they are the only alternative division algebras, and all division algebras have a real dimension of 1, 2, 4 or 8, which is denoted by  $\beta$ , see Baez [1, Theorems 1, 2 and 3]. In other branches of mathematics, the parameters  $\alpha = 2/\beta$  and  $t = \beta/4$  are used, see Edelman and Rao [12] and Kabe [20], respectively.

Let  $\mathcal{L}_{m,n}^\beta$  be the linear space of all  $m \times n$  matrices of rank  $m \leq n$  over  $\mathfrak{F}$  with  $m$  distinct positive singular values, where  $\mathfrak{F}$  denotes a *real finite-dimensional normed division algebra*. Let  $\mathfrak{F}^{m \times n}$  be the set of all  $m \times n$  matrices over  $\mathfrak{F}$ . The dimension of  $\mathfrak{F}^{m \times n}$  over  $\mathbb{R}$  is  $\beta mn$ . Let  $\mathbf{A} \in \mathfrak{F}^{m \times n}$ , then  $\mathbf{A}^* = \overline{\mathbf{A}}^T$  denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division

algebras.

Table 1: Notation

Real	Complex	Quaternion	Octonion	Generic notation
Semi-orthogonal	Semi-unitary	Semi-symplectic	Semi-exceptional type	$\mathcal{V}_{m,n}^\beta$
Orthogonal	Unitary	Symplectic	Exceptional type	$\mathfrak{U}^\beta(m)$
Symmetric	Hermitian	Quaternion hermitian	Octonion hermitian	$\mathfrak{S}_m^\beta$

In addition, let  $\mathfrak{P}_m^\beta$  be the *cone of positive definite matrices*  $\mathbf{S} \in \mathfrak{F}^{m \times m}$ ; then  $\mathfrak{P}_m^\beta$  is an open subset of  $\mathfrak{S}_m^\beta$ .

Let  $\mathfrak{D}_m^\beta$  be the *diagonal subgroup* of  $\mathcal{L}_{m,m}^\beta$  consisting of all  $\mathbf{D} \in \mathfrak{F}^{m \times m}$ ,  $\mathbf{D} = \text{diag}(d_1, \dots, d_m)$ .

For any matrix  $\mathbf{X} \in \mathfrak{F}^{n \times m}$ ,  $d\mathbf{X}$  denotes the *matrix of differentials*  $(dx_{ij})$ . Finally, we define the *measure* or volume element  $(d\mathbf{X})$  when  $\mathbf{X} \in \mathfrak{F}^{m \times n}$ ,  $\mathfrak{S}_m^\beta$ ,  $\mathfrak{D}_m^\beta$  or  $\mathcal{V}_{m,n}^\beta$ , see Dimitriu [11].

If  $\mathbf{X} \in \mathfrak{F}^{m \times n}$  then  $(d\mathbf{X})$  (the Lebesgue measure in  $\mathfrak{F}^{m \times n}$ ) denotes the exterior product of the  $\beta mn$  functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^m \bigwedge_{j=1}^n dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^\beta dx_{ij}^{(k)}.$$

If  $\mathbf{S} \in \mathfrak{S}_m^\beta$  (or  $\mathbf{S} \in \mathfrak{T}_L^\beta(m)$  is a lower triangular matrix) then  $(d\mathbf{S})$  (the Lebesgue measure in  $\mathfrak{S}_m^\beta$  or in  $\mathfrak{T}_L^\beta(m)$ ) denotes the exterior product of the  $m(m+1)\beta/2$  functionally independent variables (or denotes the exterior product of the  $m(m-1)\beta/2 + n$  functionally independent variables, if  $s_{ii} \in \mathfrak{R}$  for all  $i = 1, \dots, m$ )

$$(d\mathbf{S}) = \begin{cases} \bigwedge_{i \leq j}^m \bigwedge_{k=1}^\beta ds_{ij}^{(k)}, \\ \bigwedge_{i=1}^m ds_{ii} \bigwedge_{i < j}^m \bigwedge_{k=1}^\beta ds_{ij}^{(k)}, & \text{if } s_{ii} \in \mathfrak{R}. \end{cases}$$

The context generally establishes the conditions on the elements of  $\mathbf{S}$ , that is, if  $s_{ij} \in \mathfrak{R}$ ,  $\in \mathfrak{C}$ ,  $\in \mathfrak{H}$  or  $\in \mathfrak{D}$ . It is considered that

$$(d\mathbf{S}) = \bigwedge_{i \leq j}^m \bigwedge_{k=1}^\beta ds_{ij}^{(k)} \equiv \bigwedge_{i=1}^m ds_{ii} \bigwedge_{i < j}^m \bigwedge_{k=1}^\beta ds_{ij}^{(k)}.$$

Observe, too, that for the Lebesgue measure  $(d\mathbf{S})$  defined thus, it is required that  $\mathbf{S} \in \mathfrak{P}_m^\beta$ , that is,  $\mathbf{S}$  must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If  $\mathbf{\Lambda} \in \mathfrak{D}_m^\beta$  then  $(d\mathbf{\Lambda})$  (the Lebesgue measure in  $\mathfrak{D}_m^\beta$ ) denotes the exterior product of the  $\beta m$  functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^n \bigwedge_{k=1}^\beta d\lambda_i^{(k)}.$$

If  $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$  then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^m \bigwedge_{j=i+1}^n \mathbf{h}_j^* d\mathbf{h}_i.$$

where  $\mathbf{H} = (\mathbf{H}_1^* | \mathbf{H}_2^*)^* = (\mathbf{h}_1, \dots, \mathbf{h}_m | \mathbf{h}_{m+1}, \dots, \mathbf{h}_n)^* \in \mathfrak{U}^\beta(n)$ . It can be proved that this differential form does not depend on the choice of the  $\mathbf{H}_2$  matrix. When  $n = 1$ ;  $\mathcal{V}_{m,1}^\beta$  defines the unit sphere in  $\mathfrak{F}^m$ . This is, of course, an  $(m-1)\beta$ -dimensional surface in  $\mathfrak{F}^m$ . When  $n = m$  and denoting  $\mathbf{H}_1$  by  $\mathbf{H}$ ,  $(\mathbf{H}d\mathbf{H}^*)$  is termed the *Haar measure* on  $\mathfrak{U}^\beta(m)$ .

The surface area or volume of the Stiefel manifold  $\mathcal{V}_{m,n}^\beta$  is

$$\text{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} (\mathbf{H}_1 d\mathbf{H}_1^*) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^\beta[n\beta/2]}, \quad (1)$$

where  $\Gamma_m^\beta[a]$  denotes the multivariate *Gamma function* for the space  $\mathfrak{S}_m^\beta$ , and is defined by

$$\begin{aligned} \Gamma_m^\beta[a] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \text{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} (d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a - (i-1)\beta/2], \end{aligned}$$

where  $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$ ,  $|\cdot|$  denotes the determinant and  $\text{Re}(a) > (m-1)\beta/2$ , see Gross and Richards [16]. Similarly, from Herz [18] the *multivariate beta function* for the space  $\mathfrak{S}_m^\beta$ , can be defined as

$$\begin{aligned} \mathcal{B}_m^\beta[b, a] &= \int_{\mathbf{0} < \mathbf{B} < \mathbf{I}_m} |\mathbf{B}|^{a-(m-1)\beta/2-1} |\mathbf{I}_m - \mathbf{B}|^{b-(m+1)\beta/2-1} (d\mathbf{B}) \\ &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} |\mathbf{A}|^{a-(m-1)\beta/2-1} |\mathbf{I}_m + \mathbf{A}|^{-(a+b)} (d\mathbf{A}) \\ &= \frac{\Gamma_m^\beta[a] \Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b]}, \end{aligned} \quad (2)$$

where  $\mathbf{A} = (\mathbf{I} - \mathbf{B})^{-1} - \mathbf{I}$ ,  $\text{Re}(a) > (m-1)\beta/2$  and  $\text{Re}(b) > (m-1)\beta/2$ .

Now, we show three Jacobians in terms of the  $\beta$  parameter, which are based on the work of Kabe [20] and Dimitriu [11]. These results are proposed as extensions of real, complex or quaternion cases, see James [19], Khatri [22], Metha [26], Ratnarajah *et al.* [30] and Li and Xue [25], also see Díaz-García and Gutiérrez-Jáimez [6].

**Lemma 2.1.** *Let  $\mathbf{X}$  and  $\mathbf{Y} \in \mathcal{L}_{m,n}^\beta$ , and let  $\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{B} + \mathbf{C}$ , where  $\mathbf{A} \in \mathcal{L}_{m,m}^\beta$ ,  $\mathbf{B} \in \mathcal{L}_{n,n}^\beta$  and  $\mathbf{C} \in \mathcal{L}_{m,n}^\beta$  are constant matrices. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^* \mathbf{A}|^{\beta n/2} |\mathbf{B}^* \mathbf{B}|^{\beta m/2} (d\mathbf{X}). \quad (3)$$

**Lemma 2.2** (Singular value decomposition, *SVD*). *Let  $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$ , such that  $\mathbf{X} = \mathbf{V}^* \mathbf{D} \mathbf{W}_1$  with  $\mathbf{V} \in \mathfrak{U}^\beta(m)$ ,  $\mathbf{W}_1 \in \mathcal{V}_{m,n}^\beta$  and  $\mathbf{D} = \text{diag}(d_1, \dots, d_m) \in \mathfrak{D}_m^1$ ,  $d_1 > \dots > d_m > 0$ . Then*

$$(d\mathbf{X}) = 2^{-m} \pi^\tau \prod_{i=1}^m d_i^{\beta(n-m+1)-1} \prod_{i < j}^m (d_i^2 - d_j^2)^\beta (d\mathbf{D}) (\mathbf{V} d\mathbf{V}^*) (\mathbf{W}_1 d\mathbf{W}_1^*), \quad (4)$$

where

$$\tau = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

**Lemma 2.3.** *Let  $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$ , and  $\mathbf{S} = \mathbf{X}\mathbf{X}^* \in \mathfrak{P}_m^\beta$ . Then*

$$(d\mathbf{X}) = 2^{-m} |\mathbf{S}|^{\beta(n-m+1)/2-1} (d\mathbf{S}) (\mathbf{V}_1 d\mathbf{V}_1^*). \quad (5)$$

### 3 Matricvariate Pearson type II distribution

In the real case, the *matricvariate Pearson type II distribution* (also known in the literature as **matricvariate inverted T distribution**) was studied in detail by Dickey [10] and Cadet [3], see also Press [29]. This distribution was previously studied by Khatri [21], also in the real case.

**Theorem 3.1.** Let  $\mathbf{R} \in \mathcal{L}_{m,n}^\beta$  defined as

$$\mathbf{R} = \mathbf{L}^{-1} \mathbf{X}$$

where  $\mathbf{L}$  is any square root of  $\mathbf{U} = (\mathbf{U}_1 + \mathbf{X}\mathbf{X}^*)$  such that  $\mathbf{L}\mathbf{L}^* = \mathbf{U}$ ,  $\mathbf{U}_1 \sim \mathcal{W}_m^\beta(\nu, \mathbf{I})$ ,  $\nu > \beta(m-1)$ , independent of  $\mathbf{X} \sim \mathcal{N}_{m \times n}^\beta(\mathbf{0}, \mathbf{I}_m \otimes \mathbf{I})$ , and  $\mathbf{I} \in \mathfrak{P}_m^\beta$ . Then  $\mathbf{U} \sim \mathcal{W}_m^\beta(\nu+n, \mathbf{I})$  independently of  $\mathbf{R}$ . Furthermore, the density of  $\mathbf{R}$  is

$$\frac{\Gamma_m^\beta[\beta(n+\nu)/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[\beta\nu/2]} |\mathbf{I}_m - \mathbf{R}\mathbf{R}^*|^{\beta(\nu-m+1)/2-1}, \quad \mathbf{I}_m - \mathbf{R}\mathbf{R}^* \in \mathfrak{P}_m^\beta \quad (6)$$

which is termed the *matricvariate Pearson type II distribution*<sup>1</sup>.

*Proof.* From Kabe [20] and Díaz-García and Gutiérrez-Jáimez [6, 9] the joint density of  $\mathbf{U}_1$  and  $\mathbf{X}$  is

$$\propto |\mathbf{U}_1|^{\beta(\nu-m+1)/2-1} \text{etr}\{-\beta\mathbf{I}^{-1}(\mathbf{U}_1 + \mathbf{X}\mathbf{X}^*)/2\},$$

where the constant of proportionality is

$$c = \frac{1}{(2\beta^{-1})^{\beta m\nu/2}\Gamma_m^\beta[\beta\nu/2]|\mathbf{I}|^{\beta\nu/2}} \cdot \frac{1}{(2\pi\beta^{-1})^{\beta mn/2}|\mathbf{I}|^{\beta n/2}}.$$

Making the change of variable  $\mathbf{U}_1 = (\mathbf{U} - \mathbf{X}\mathbf{X}^*)$  and  $\mathbf{X} = \mathbf{L}\mathbf{R}$ , where  $\mathbf{U} = \mathbf{L}\mathbf{L}^*$ , then by (3)

$$(d\mathbf{U}_1)(d\mathbf{X}) = |\mathbf{L}\mathbf{L}^*|^{\beta n/2}(d\mathbf{U})(d\mathbf{R}) = |\mathbf{U}|^{\beta n/2}(d\mathbf{U})(d\mathbf{R}),$$

and observing that  $|\mathbf{U}_1| = |\mathbf{U} - \mathbf{X}\mathbf{X}^*| = |\mathbf{U} - \mathbf{L}\mathbf{R}\mathbf{R}^*\mathbf{L}^*| = |\mathbf{U}||\mathbf{I}_m - \mathbf{R}\mathbf{R}^*|$ , the joint density of  $\mathbf{U}$  and  $\mathbf{R}$  is

$$\propto |\mathbf{I}_m - \mathbf{R}\mathbf{R}^*|^{\beta(\nu-m+1)/2-1} |\mathbf{U}|^{\beta(\nu+n-m+1)/2-1} \text{etr}\{-\beta\mathbf{I}^{-1}\mathbf{U}/2\}. \quad \square$$

□

Similarly to Dickey [10], (6) is alternatively given by

$$\frac{\Gamma_n^\beta[\beta(n+\nu)/2]}{\pi^{mn\beta/2}\Gamma_n^\beta[\beta(n+\nu-m)/2]} |\mathbf{I}_n - \mathbf{R}^*\mathbf{R}|^{\beta(\nu-m+1)/2-1}, \quad \mathbf{I}_n - \mathbf{R}^*\mathbf{R} \in \mathfrak{P}_n^\beta. \quad (7)$$

**Corollary 3.1.** Let  $\mathbf{Q} = (\mathbf{M}^*)^{-1}\mathbf{R}\mathbf{N}^{-1} + \boldsymbol{\mu}$ ,  $\mathbf{R}$  as in Theorem 3.1,  $\mathbf{M}$  and  $\mathbf{N}$  are any square root of the constant matrices  $\mathbf{B} = \mathbf{M}\mathbf{M}^* \in \mathfrak{P}_m^\beta$  and  $\mathcal{A} = \mathbf{N}\mathbf{N}^* \in \mathfrak{P}_n^\beta$ , respectively, and  $\boldsymbol{\mu} \in \mathcal{L}_{m,n}^\beta$  is constant. Then, from (6) and (7) the density of  $\mathbf{Q}$  is given by

$$\frac{\Gamma_m^\beta[\beta(n+\nu)/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[\beta\nu/2]} \frac{|\mathbf{B}|^{\beta(\nu+n-m+1)/2-1}}{|\mathcal{A}|^{\beta m/2}} |\mathbf{B}^{-1} - (\mathbf{Q} - \boldsymbol{\mu})\mathcal{A}^{-1}(\mathbf{Q} - \boldsymbol{\mu})^*|^{\beta(\nu-m+1)/2-1},$$

<sup>1</sup>In the literature it is customary to use the real matricvariate Pearson type II distribution, complex matricvariate Pearson type II distribution, quaternion matricvariate Pearson type II distribution and octonion matricvariate Pearson type II distribution; here, however, we use simply matricvariate Pearson type II distribution as the generic term.

$$\mathbf{B}^{-1} - (\mathbf{Q} - \boldsymbol{\mu})\mathcal{A}^{-1}(\mathbf{Q} - \boldsymbol{\mu})^* \in \mathfrak{P}_m^\beta,$$

and as

$$\frac{\Gamma_n^\beta[\beta(n + \nu)/2]}{\pi^{mn\beta/2}\Gamma_n^\beta[\beta(\nu + n - m)/2]} \frac{|\mathbf{B}|^{\beta\nu/2}}{|\mathcal{A}|^{\beta(\nu+1)/2-1}} |\mathcal{A} - (\mathbf{Q} - \boldsymbol{\mu})^* \mathbf{B}(\mathbf{Q} - \boldsymbol{\mu})|^{\beta(\nu-m+1)/2-1},$$

$$\mathcal{A} - (\mathbf{Q} - \boldsymbol{\mu})^* \mathbf{B}(\mathbf{Q} - \boldsymbol{\mu}) \in \mathfrak{P}_n^\beta.$$

This fact shall denoted as  $\mathbf{Q} \sim \mathcal{PII}_{m \times n}^\beta(\nu, \boldsymbol{\mu}, \mathbf{B}, \mathcal{A})$  (and of course  $\mathbf{R} \sim \mathcal{PII}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ ).

*Proof.* The proof follows observing that, by (3)

$$(d\mathbf{R}) = |\mathbf{M}\mathbf{M}^*|^{\beta n/2} |\mathbf{N}\mathbf{N}^*|^{\beta m/2} (d\mathbf{Q}) = |\mathbf{B}|^{\beta n/2} |\mathcal{A}|^{\beta m/2} (d\mathbf{Q})$$

and that

$$|\mathbf{B}^{-1} - (\mathbf{Q} - \boldsymbol{\mu})\mathcal{A}^{-1}(\mathbf{Q} - \boldsymbol{\mu})^*| = |\mathbf{B}|^{-1} |\mathcal{A}|^{-1} |\mathcal{A} - (\mathbf{Q} - \boldsymbol{\mu})^* \mathbf{B}(\mathbf{Q} - \boldsymbol{\mu})|. \quad \square$$

□

**Corollary 3.2.** Assume that  $\mathbf{Q} \sim \mathcal{PII}_{m \times n}^\beta(\nu, \boldsymbol{\mu}, \mathbf{B}, \mathcal{A})$ , then

$$\mathbf{Q}^* \sim \mathcal{PII}_{n \times m}^\beta(\nu, \boldsymbol{\mu}^*, \mathcal{A}^{-1}, \mathbf{B}^{-1}).$$

*Proof.* The proof follows immediately from the two expressions for the density of  $\mathbf{Q}$  in Corollary 3.1. □

**Corollary 3.3.** Let  $\mathbf{R} \in \mathcal{L}_{m,n}^\beta$  defined as

$$\mathbf{R} = \mathbf{Y}\mathbf{L}_1^{-1}$$

where  $\mathbf{L}_1$  is any square root of  $\mathbf{V} = (\mathbf{V}_1 + \mathbf{Y}^*\mathbf{Y})$  such that  $\mathbf{L}_1\mathbf{L}_1^* = \mathbf{V}$ ,  $\mathbf{V}_1 \sim \mathcal{W}_n^\beta(\nu + n - m, \mathbf{J})$ , independent of  $\mathbf{Y} \sim \mathcal{N}_{m \times n}^\beta(\mathbf{0}, \mathbf{J} \otimes \mathbf{I}_n)$ , and  $\mathbf{J} \in \mathfrak{P}_n^\beta$ . Then  $\mathbf{V} \sim \mathcal{W}_n^\beta(\nu + n, \mathbf{J})$  independently of  $\mathbf{R}$ . Furthermore,  $\mathbf{R} \sim \mathcal{PII}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ .

*Proof.* The proof is a verbatim copy of the proof of Theorem 3.1. □

Now, assume that  $\mathbf{R} \sim \mathcal{PII}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$  with  $n \geq m$  and let  $\mathbf{B} \in \mathfrak{P}_m^\beta$  defined as  $\mathbf{B} = \mathbf{R}\mathbf{R}^*$  then, under the conditions of Theorem 3.1 and Corollary 3.3, we have

$$\begin{aligned} \mathbf{B} &= \mathbf{L}^{-1}\mathbf{X}\mathbf{X}^*(\mathbf{L}^{-1})^* = \mathbf{L}^{-1}\mathbf{W}(\mathbf{L}^{-1})^* \\ &= \mathbf{Y}(\mathbf{V}_1 + \mathbf{Y}^*\mathbf{Y})^{-1}\mathbf{Y}^*, \end{aligned}$$

where  $\mathbf{W} = \mathbf{X}\mathbf{X}^* \sim \mathcal{W}_m^\beta(n, \mathbf{I})$ ,  $n > \beta(m - 1)$ . Thus:

**Theorem 3.2.** The density of  $\mathbf{B}$  is

$$\frac{1}{\mathcal{B}_m^\beta[\beta\nu/2, \beta n/2]} |\mathbf{B}|^{\beta(n-m+1)/2-1} |\mathbf{I}_m - \mathbf{B}|^{\beta(\nu-m+1)/2-1}, \quad (8)$$

where  $\mathcal{B}_m^\beta[\cdot, \cdot]$  is given by (2) and  $\mathbf{B}$  is said to have a matricvariate beta type I distribution.

*Proof.* The proof follows from (6) by applying (1) and (5). □

□

In addition, assume that  $n < m$  and let  $\tilde{\mathbf{B}} \in \mathfrak{P}_n^\beta$  defined as  $\tilde{\mathbf{B}} = \mathbf{R}^* \mathbf{R}$  then, under the conditions of Theorem 3.1 and Corollary 3.3 we have

$$\begin{aligned}\tilde{\mathbf{B}} &= \mathbf{X}^*(\mathbf{U}_1 + \mathbf{X}^* \mathbf{X})^{-1} \mathbf{X} \\ &= \mathbf{L}_1^{-1} \mathbf{Y}^* \mathbf{Y} (\mathbf{L}_1^{-1})^* = \mathbf{L}_1^{-1} \mathbf{W}_1 (\mathbf{L}_1^{-1})^*\end{aligned}$$

where  $\mathbf{W}_1 = \mathbf{Y}^* \mathbf{Y} \sim \mathcal{W}_n^\beta(m, \mathbf{I})$ ,  $m > \beta(n-1)$ , Thus:

**Theorem 3.3.**  $\tilde{\mathbf{B}}$  has the density

$$\frac{1}{\mathcal{B}_n^\beta[\beta(\nu + n - m)/2, \beta m/2]} |\tilde{\mathbf{B}}|^{\beta(m-n+1)/2-1} |\mathbf{I}_n - \tilde{\mathbf{B}}|^{\beta(\nu-m+1)/2-1}. \quad (9)$$

Also, we say that  $\tilde{\mathbf{B}}$  has a matrixvariate beta type I distribution.

*Proof.* The proof is the same as that given in Theorem 3.2. Alternatively, observe that density (9) can be obtained from density (8) making the following substitutions, see Muirhead [28, Eq. (7), p. 455] and Srivastava & Khatri [31, p. 96],

$$m \rightarrow n, \quad n \rightarrow m, \quad \nu \rightarrow \nu + n - m. \quad \square \quad (10)$$

□

Densities (8) and (9) have been studied by several authors in the real case, see Khatri [23] and Srivastava & Khatri [31], Cadet [3], among many others; and by James [19], Díaz-García and Gutiérrez-Jáimez [7] and Díaz-García and Gutiérrez-Jáimez [5] and Díaz-García and Gutiérrez-Jáimez [8], in noncentral, doubly noncentral, singular and nonsingular and complex cases, among many other authors. By the analogous construction of  $\mathbf{B} = \mathbf{R} \mathbf{R}^*$  (or  $\tilde{\mathbf{B}} = \mathbf{R}^* \mathbf{R}$ ) in terms of  $\mathbf{R}$  compared with the construction of the Wishart matrix in terms of matrix multivariate normal distribution, the distribution of  $\mathbf{B} = \mathbf{R} \mathbf{R}^*$  (or  $\tilde{\mathbf{B}} = \mathbf{R}^* \mathbf{R}$ ) is sometimes termed the studentised Wishart distribution. Kabe [20] studied densities (8) and (9) for the hypercomplex case.

## 4 Matrix multivariate Pearson type II distribution

**Theorem 4.1.** Let  $(S^{1/2})^2 \sim \chi^{2,\beta}(\nu)$  independent of  $\mathbf{Y} \sim \mathcal{N}_{m \times n}^\beta(\mathbf{0}, \mathbf{I}_m \otimes \mathbf{I}_n)$ , and define  $S_1 = S + \text{tr } \mathbf{Y} \mathbf{Y}^*$  and  $\mathbf{R}_1 = S_1^{-1/2} \mathbf{Y}$ . Then  $S_1 \sim \chi^2(\nu + mn)$  independently of  $\mathbf{R}_1$ . Furthermore, the density of  $\mathbf{R}_1$  is

$$\frac{\Gamma_1^\beta[\beta(\nu + mn)/2]}{\pi^{\beta mn/2} \Gamma_1^\beta[\beta\nu/2]} (1 - \text{tr } \mathbf{R}_1 \mathbf{R}_1^*)^{\beta\nu/2-1}, \quad 1 - \text{tr } \mathbf{R}_1 \mathbf{R}_1^* > 0, \quad (11)$$

which is termed the matrix multivariate Pearson type II distribution.

*Proof.* The joint density of  $S$  and  $\mathbf{Y}$  is

$$\propto s^{\beta\nu/2-1} \text{etr}\{-\beta(s + \text{tr } \mathbf{Y} \mathbf{Y}^*)/2\},$$

and the desired results are obtained analogously to the proof of Theorem 3.1. □ □

**Corollary 4.1.** Let  $\mathbf{Q}_1 = (\mathbf{M}^*)^{-1} \mathbf{R}_1 \mathbf{N}^{-1} + \boldsymbol{\mu}$ ,  $\mathbf{R}_1$  as in Theorem 3.3,  $\mathbf{M}$  and  $\mathbf{N}$  are any square root of the constant matrices  $\mathbf{B} = \mathbf{M} \mathbf{M}^* \in \mathfrak{P}_m^\beta$  and  $\mathbf{D} = \mathbf{N} \mathbf{N}^* \in \mathfrak{P}_n^\beta$ , respectively, and  $\boldsymbol{\mu} \in \mathcal{L}_{m,n}^\beta$  is constant. Then,

$$\frac{\Gamma_1^\beta[\beta(\nu + mn)/2]}{\pi^{\beta mn/2} \Gamma_1^\beta[\beta\nu/2]} |\mathbf{B}|^{\beta n/2} |\mathbf{D}|^{\beta m/2} [1 - \text{tr } \mathbf{B}(\mathbf{Q}_1 - \boldsymbol{\mu}) \mathbf{D}(\mathbf{Q}_1 - \boldsymbol{\mu})^*]^{\beta\nu/2-1},$$

$$1 - \text{tr } \mathbf{B}(\mathbf{Q}_1 - \boldsymbol{\mu})\mathcal{A}(\mathbf{Q}_1 - \boldsymbol{\mu})^* > 0.$$

Hence, we write

$$\mathbf{Q}_1 \sim \mathcal{MPII}_{m \times n}^\beta(\nu, \boldsymbol{\mu}, \mathbf{B}, \mathcal{A}),$$

and, in particular  $\mathbf{R}_1 \sim \mathcal{MPII}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ .

*Proof.* The proof follows observing that, by (3)

$$(d\mathbf{R}_1) = |\mathbf{M}\mathbf{M}^*|^{\beta n/2} |\mathbf{N}\mathbf{N}^*|^{\beta m/2} (d\mathbf{Q}_1) = |\mathbf{B}|^{\beta n/2} |\mathcal{A}|^{\beta m/2} (d\mathbf{Q}_1). \quad \square$$

□

Now, assuming that  $\mathbf{R}_1 \sim \mathcal{PII}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ , with  $n \geq m$  and defining  $\mathbf{B}_1 = \mathbf{R}_1 \mathbf{R}_1^* \in \mathfrak{P}_m^\beta$ , then, under the conditions of Theorem 4.1 we have that

$$\mathbf{B}_1 = \mathbf{S}^{-1} \mathbf{Y} \mathbf{Y}^*$$

where  $\mathbf{W} = \mathbf{Y} \mathbf{Y}^* \sim \mathcal{W}_m^\beta(n, \mathbf{I}_m)$ ,  $n > \beta(m-1)$ . Furthermore:

**Theorem 4.2.** *The density of  $\mathbf{B}_1$  is*

$$\frac{\Gamma_1^\beta[\beta(\nu + mn)/2]}{\Gamma_1^\beta[\beta\nu/2] \Gamma_m^\beta[\beta n/2]} |\mathbf{B}_1|^{\beta(n-m+1)/2-1} (1 - \text{tr } \mathbf{B}_1)^{\beta\nu/2-1}, \quad \mathbf{0} < \mathbf{B}_1 < \mathbf{I}_m, \quad (12)$$

$\mathbf{B}_1$  is said to have a matrix multivariate beta type I distribution.

*Proof.* The proof follows from (11) by applying (1) and (5). □ □

Similarly, if  $n < m$  and  $\tilde{\mathbf{B}}_1 = \mathbf{R}_1^* \mathbf{R}_1 \in \mathfrak{P}_n^\beta$ , thus:

**Theorem 4.3.**  *$\tilde{\mathbf{B}}_1$  has the density*

$$\frac{\Gamma_1^\beta[\beta(\nu + mn)/2]}{\Gamma_1^\beta[\beta\nu/2] \Gamma_n^\beta[\beta m/2]} |\tilde{\mathbf{B}}_1|^{\beta(m-n+1)/2-1} (1 - \text{tr } \tilde{\mathbf{B}}_1)^{\beta\nu/2-1}, \quad \mathbf{0} < \tilde{\mathbf{B}}_1 < \mathbf{I}_n. \quad (13)$$

Thus,  $\tilde{\mathbf{B}}_1$  is said to have a matrix multivariate distribution type I distribution.

*Proof.* Density (13) can be obtained from density (12) by making the following substitutions,

$$m \rightarrow n, \quad n \rightarrow m. \quad \square \quad (14)$$

□

## 5 Singular value densities

In this section, we derive the joint density of the singular values of matrices  $\mathbf{R}$ ,  $\tilde{\mathbf{R}}$ ,  $\mathbf{R}_1$  and  $\tilde{\mathbf{R}}_1$ . In addition, the joint densities of the eigenvalues of  $\mathbf{B}$ ,  $\tilde{\mathbf{B}}$ ,  $\mathbf{B}_1$  and  $\tilde{\mathbf{B}}_1$  are obtained.

**Theorem 5.1.** *Let  $\delta_1, \dots, \delta_m$  be the singular values of  $\mathbf{R} \sim \mathcal{PII}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ ,  $1 > \delta_1 > \dots > \delta_m > 0$ . Then its joint density is*

$$\frac{2^m \pi^{\beta m^2 + \tau}}{\Gamma_m^\beta[\beta m/2] \mathcal{B}_m^\beta[\beta\nu/2, \beta n/2]} \prod_{i=1}^m \delta_i^{\beta(n-m+1)-1} (1 - \delta_i^2)^{\beta(\nu-m+1)/2-1} \prod_{i < j}^m (\delta_i^2 - \delta_j^2)^\beta \quad (15)$$

where  $\tau$  is defined in Lemma 2.2.



*Proof.* The proof follows immediately from (6), first using (4) and then applying (1).  $\square$   $\square$

The joint density of the singular values of  $\tilde{\mathbf{R}}$  is obtained from (15) after making the substitutions (10).

**Theorem 5.2.** *Suppose that  $\mathbf{R}_1 \sim \mathcal{MPII}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$  and let  $\alpha_1, \dots, \alpha_m, 1 > \alpha_1 > \dots > \alpha_m > 0, 0 < \sum_{i=1}^m \alpha_i < 1$ , its singular values. Then its joint density is*

$$\frac{2^m \pi^{\beta m^2/2 + \tau} \Gamma_1^\beta[\beta(\nu + mn)/2]}{\Gamma_1^\beta[\beta\nu/2] \Gamma_m^\beta[\beta m/2] \Gamma_m^\beta[\beta n/2]} \left(1 - \sum_{i=1}^m \alpha_i^2\right)^{\beta\nu/2-1} \prod_{i=1}^m \alpha_i^{\beta(n-m+1)-1} \prod_{i < j}^m (\alpha_i^2 - \alpha_j^2)^\beta \quad (16)$$

*Proof.* The proof is analogous to that given for Theorem 5.1.  $\square$   $\square$

Similarly, the joint density of the singular values of  $\tilde{\mathbf{R}}_1$  is obtained from (16) and making the substitutions (14).

Finally, observe that  $\delta_i = \sqrt{\text{eig}_i(\mathbf{R}\mathbf{R}^*)}$  and  $\alpha_i = \sqrt{\text{eig}_i(\mathbf{R}_1\mathbf{R}_1^*)}$ , where  $\text{eig}_i(\mathbf{A})$ ,  $i = 1, \dots, m$ , denotes the  $i$ -th eigenvalue of  $\mathbf{A}$ . Let  $\lambda_i = \text{eig}_i(\mathbf{R}\mathbf{R}^*)$  and  $\gamma_i = \text{eig}_i(\mathbf{R}_1\mathbf{R}_1^*)$ , hence, observing that, for example,  $\delta_i = \sqrt{\lambda_i}$  and then

$$\bigwedge_{i=1}^m d\delta_i = \bigwedge_{i=1}^m 2^{-m} \prod_{i=1}^m \lambda_i^{-1/2} d\lambda_i,$$

the corresponding joint density of  $\lambda_1, \dots, \lambda_m, 1 > \lambda_1 > \dots > \lambda_m > 0$  is obtained from (15) as

$$\frac{\pi^{\beta m^2 + \tau}}{\Gamma_m^\beta[\beta m/2] \mathcal{B}_m^\beta[\beta\nu/2, \beta n/2]} \prod_{i=1}^m \lambda_i^{\beta(n-m+1)/2-1} (1 - \lambda_i)^{\beta(\nu-m+1)/2-1} \prod_{i < j}^m (\lambda_i - \lambda_j)^\beta.$$

Similarly, the joint density of  $\gamma_1, \dots, \gamma_m, 1 > \gamma_1 > \dots > \gamma_m > 0, 0 < \sum_{i=1}^m \alpha_i < 1$ , is obtained from (16) as

$$\frac{\pi^{\beta m^2/2 + \tau} \Gamma_1^\beta[\beta(\nu + mn)/2]}{\Gamma_1^\beta[\beta\nu/2] \Gamma_m^\beta[\beta m/2] \Gamma_m^\beta[\beta n/2]} \left(1 - \sum_{i=1}^m \gamma_i\right)^{\beta\nu/2-1} \prod_{i=1}^m \gamma_i^{\beta(n-m+1)/2-1} \prod_{i < j}^m (\gamma_i - \gamma_j)^\beta.$$

## Conclusions

- Beyond a doubt, in any generalisation of results there is a price to be paid, and in this case the price is that of acquiring a basic understanding of some concepts of abstract algebra, which can initially be summarised as the use of notation and a basic minimum set of definitions. However, we believe that a detailed study of mathematical properties from a statistical standpoint can have a potential impact on statistical theory.
- Furthermore, note that  $\mathbf{X} \in \mathfrak{L}_{m,n}^\beta$  has a matrix multivariate elliptically contoured distribution for real normed division algebras if its density, with respect to the Lebesgue measure, is given by (see Díaz-García and Gutiérrez-Jáimez [6]):

$$\frac{C^\beta(m, n)}{|\Sigma|^{\beta n/2} |\Theta|^{\beta m/2}} h \left\{ \text{tr} \left[ \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu}) \Theta^{-1} (\mathbf{X} - \boldsymbol{\mu})^* \right] \right\}, \quad (17)$$

where  $\boldsymbol{\mu} \in \mathfrak{L}_{m,n}^\beta$ ,  $\Sigma \in \mathfrak{P}_m^\beta$ ,  $\Theta \in \mathfrak{P}_m^\beta$ . The function  $h : \mathfrak{F} \rightarrow [0, \infty)$  is termed the generator function, and it is such that  $\int_{\mathfrak{P}_1^\beta} u^{\beta nm-1} h(u^2) du < \infty$  and

$$C^\beta(m, n) = \frac{\Gamma[\beta mn/2]}{2\pi^{\beta mn/2}} \left\{ \int_{\mathfrak{P}_1^\beta} u^{\beta nm-1} h(u^2) du \right\}$$

Such a distribution is denoted by  $\mathbf{X} \sim \mathcal{E}_{n \times m}^\beta(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Theta}, h)$ , for the real case, see Fang and Zhang [14] and Gupta, and Varga [17]; and Micheas *et al.* [27] for the complex case. Observe that this class of matrix multivariate distributions includes normal, contaminated normal, Pearson type II and VII, Kotz, Jensen-Logistic, power exponential and Bessel distributions, among others; these distributions have tails that are more or less weighted, and/or present a greater or smaller degree of kurtosis than the normal distribution.

Assume that  $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \vdots & \mathbf{X}_2 \\ m \times n & & m \times \nu \end{pmatrix} \sim \mathcal{E}_{m \times n + \nu}^\beta(\mathbf{0}, \mathbf{I}_m, \mathbf{I}_{n+\nu}, h)$ ,  $n, \nu \geq m$ ; and define,  $\mathbf{R} = \mathbf{L}^{-1}\mathbf{X}_1$ , where  $\mathbf{L}$  is any square root of  $\mathbf{V} = (\mathbf{X}_2\mathbf{X}_2^* + \mathbf{X}_1\mathbf{X}_1^*)$  such that  $\mathbf{L}\mathbf{L}^* = \mathbf{V}$ . Then  $\mathbf{R} \sim \mathcal{PII}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$  independently of  $\mathbf{V} \sim \mathcal{GW}_m^\beta(n + \nu, \mathbf{I}_m, h)$ ,  $n + \nu > \beta(m - 1)$ , where  $\mathcal{GW}_m^\beta(\cdot)$  denotes the generalised Wishart distribution based on an elliptical distribution, see Díaz-García and Gutiérrez-Jáimez [6] and Díaz-García and Gutiérrez-Jáimez [9]. From (17) the density of  $\mathbf{X}$  is

$$C^\beta(m, n + \nu)h \{ \text{tr}(\mathbf{X}_1\mathbf{X}_1^* + \mathbf{X}_2\mathbf{X}_2^*) \}.$$

Let  $\mathbf{V}_0 = \mathbf{X}_2\mathbf{X}_2^*$  then by (2.3),  $(d\mathbf{X}_2) = 2^{-m}|\mathbf{V}_0|^{\beta(\nu-m+1)/2-1}(d\mathbf{V}_0)(\mathbf{H}_1 d\mathbf{H}_1^*)$ . Thus, the marginal density of  $\mathbf{X}_1$  and  $\mathbf{V}_0$  is obtained by integrating over  $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$ , and then using (1), to obtain

$$\frac{C^\beta(m, n + \nu)\pi^{\beta\nu m/2}}{\Gamma_m^\beta[\beta\nu/2]}|\mathbf{V}_0|^{\beta(\nu-m+1)/2-1}h \{ \text{tr}(\mathbf{X}_1\mathbf{X}_1^* + \mathbf{V}_0) \}.$$

Now, let  $\mathbf{V} = (\mathbf{V}_0 + \mathbf{X}_1\mathbf{X}_1^*)$  and  $\mathbf{R} = \mathbf{L}^{-1}\mathbf{X}_1$ , where  $\mathbf{L}\mathbf{L}^* = \mathbf{V}$ , then by (2.1)

$$(d\mathbf{X}_1)(d\mathbf{V}_0) = |\mathbf{V}|^{\beta n/2}(d\mathbf{R})(d\mathbf{V}).$$

Observing that  $|\mathbf{V}_0| = |\mathbf{V}||\mathbf{I}_m - \mathbf{R}\mathbf{R}^*|$ , and so the joint density of  $\mathbf{R}$  and  $\mathbf{V}$  is

$$\frac{C^\beta(m, n + \nu)\pi^{\beta\nu m/2}}{\Gamma_m^\beta[\beta\nu/2]}|\mathbf{I}_m - \mathbf{R}\mathbf{R}^*|^{\beta(\nu-m+1)/2-1}|\mathbf{V}|^{\beta(n+\nu-m+1)/2-1}h \{ \text{tr} \mathbf{V} \}.$$

from where the desired result follows.  $\square$

Observe that in this case,  $\mathbf{X}_1$  and  $\mathbf{X}_2$  (or  $\mathbf{V}_0 = \mathbf{X}_2\mathbf{X}_2^*$ ) are stochastically dependent. Furthermore, note that only when the particular matrix multivariate elliptical distribution is the matrix multivariate normal distribution, are  $\mathbf{X}_1$  and  $\mathbf{X}_2$  (or  $\mathbf{V}_0 = \mathbf{X}_2\mathbf{X}_2^*$ ) independent. Then, we can say that the matrix multivariate Pearson type II distribution is invariant under the family of matrix multivariate elliptical distributions, and furthermore, its density is the same as when normality is assumed. In the same way, it can be proved that the matrix multivariate Pearson type II, matrix multivariate and matrix multivariate beta type I distributions are invariant under the family of matrix multivariate elliptical distributions.

- Finally, following Kabe [20], the distributions studied in this paper are easily extended to the hypercomplex case (biquaternion and bioctonion cases, which are a Jordan algebras), simply replacing  $\beta$  by  $2\beta$  in the obtained results.

## Acknowledgements

This research work was partially supported by CONACYT-México, Research Grant No. 81512 and IDI-Spain, Grants No. FQM2006-2271 and MTM2008-05785. This paper was written during J. A. Díaz-García's stay as a visiting professor at the Department of Statistics and O. R. of the University of Granada, Spain.

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